Gröbner bases of nested configurations

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Abstract

In this paper we introduce a new and large family of configurations whose toric ideals possess quadratic Gröbner bases. As an application, a generalization of algebras of Segre–Veronese type will be studied.

1 Introduction

Let $K[\mathbf{t}] = K[t_1, \ldots, t_d]$ be the polynomial ring over a field K. A finite set A of monomials of $K[\mathbf{t}]$ is called a *configuration* of $K[\mathbf{t}]$ if there exists a nonnegative vector $(w_1, \ldots, w_d) \in \mathbb{R}^d_{\geq 0}$ such that $\sum_{i=1}^d w_i a_i = 1$ for all $t_1^{a_1} \cdots t_d^{a_d} \in A$. Let A be a configuration of $K[\mathbf{t}]$. We associate A with the homogeneous semigroup ring $K[A]$ which is the subalgebra of $K[\mathbf{t}]$ generated by the monomials of A. Let $K[X] = K[\{x_M \mid M \in A\}]$ denote the polynomial ring over K in the variables x_M with $M \in A$, where each $\deg(x_M) = 1$. The *toric ideal* I_A of A is the kernel of the surjective homomorphism $\pi : K[X] \to K[A]$ defined by setting $\pi(x_M) = M$ for all $M \in A$. It is known that the toric ideal I_A is generated by the binomials $u - v$, where u and v are monomials of $K[X]$, with $\pi(u) = \pi(v)$. Moreover, since A is a configuration, I_A is generated by homogeneous binomials. See, e.g., [Stu, Section 4].

A fundamental question in commutative algebra is to determine whether $K[A]$ is *Koszul*. A Gröbner basis $\mathcal{G}$ is called a *quadratic Gröbner basis* if $\mathcal{G}$ consists of quadratic homogeneous polynomials. Even though it is difficult to prove that $K[A]$ is Koszul, the hierarchy (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) is known among the following properties:

(i) I_A possesses a quadratic Gröbner basis.

(ii) $K[A]$ is Koszul;

(iii) I_A is generated by quadratic binomials.

However both (ii) $\Longrightarrow$ (i) and (iii) $\Longrightarrow$ (ii) are false in general. One can find counterexamples for them in [OH1, Examples 2.1 and 2.2].

Let A be a configuration of a polynomial ring $K[\mathbf{t}] = K[t_1, \ldots, t_d]$ with d variables. For each $i = 1, 2, \ldots, d$, let $B_i = \{m_1^{(i)}, \ldots, m_{\lambda_i}^{(i)}\}$ be a configuration of a polynomial ring

$K[\mathbf{u}^{(i)}] = K[u_1^{(i)}, \ldots, u_{\mu_i}^{(i)}]$ with μ_i variables. The *nested configuration* arising from A and $B_1, ..., B_d$ is the configuration

$$A(B_1, \ldots, B_d) := \left\{ m_{j_1}^{(i_1)} \cdots m_{j_r}^{(i_r)} \;\middle|\; 1 \le r \in \mathbb{Z},\ t_{i_1} \cdots t_{i_r} \in A,\ \ 1 \le j_k \le \lambda_{i_k} \right\}$$

of a polynomial ring $K[\mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(d)}]$. The main result of the present paper is as follows:

Theorem 3.6. *Work with the same notation as above. If the toric ideals I_A, $I_{B_1}, \ldots, I_{B_d}$ possess quadratic Gröbner bases, then the toric ideal $I_{A(B_1,\ldots,B_d)}$ possesses a quadratic Gröbner basis.*

In addition, in Section 4, we study a quadratic Gröbner basis of the toric ideal of $A(B_1, \ldots, B_d)$ where A comes from a Segre–Veronese configuration.

2 Motivation from statistics

In this section we present a statistical problem, which motivates a generalization of Segre-Veronese configuration considered in our previous paper [AHOT].

We consider nested selection of groups and items from the groups. For example, consider an examination on mathematics which consists of J groups of problems (e.g. algebra, geometry and statistics) and each group j consists of m_j individual problems. For simplicity let $J = 3$ and $m_j \equiv 3$. Label the nine individual problems as A1, A2, A3, G1, G2, G3 and S1, S2, S3. Suppose that each examinee is asked to choose two groups of problems and then $c_j = 2$ problems from each chosen group j. Then there are 27 patterns of selections of four problems as (A1,A2,G1,G2), (A1,A3,G1,G2), (A2,A3,G1,G2), …,(G2,G3,S2,S3). Now as a simple statistical model suppose that each problem is chosen according to its own attractiveness, independent of the choices of other problems within the same group as well as the choice of other group. Let $q_{\mathrm{A1}}, q_{\mathrm{A2}}, \ldots, q_{\mathrm{S3}}$ denote the attractiveness of each problem. Then the probabilities of the selections are expressed as

$$\mathrm{Prob(A1, A2, G1, G2)} = c\, q_{\mathrm{A1}} q_{\mathrm{A2}} q_{\mathrm{G1}} q_{\mathrm{G2}},$$
$$\cdots$$
$$\mathrm{Prob(G2, G3, S2, S3)} = c\, q_{\mathrm{G2}} q_{\mathrm{G3}} q_{\mathrm{S2}} q_{\mathrm{S3}},$$

where c is the normalizing constant so that the 27 probabilities sum to one. Now associate a configuration A to the semigroup ring

$$K(q_{\mathrm{A1}} q_{\mathrm{A2}} q_{\mathrm{G1}} q_{\mathrm{G1}}, \ldots, q_{\mathrm{G2}} q_{\mathrm{G3}} q_{\mathrm{S2}} q_{\mathrm{S31}}),$$

which is a subring of the polynomial ring $K(q_{\mathrm{A1}}, q_{\mathrm{A2}}, \ldots, q_{\mathrm{S3}})$ in nine variables. A system of generators of the toric ideal for I_A, such as the reduced Gröbner basis, is required for statistical test of this model. This example corresponds to $A(B_1, B_2, B_3)$ where $A = \{t_1t_2, t_1t_3, t_2t_3\}$ and B_1, B_2, B_3 are copies of A with different variables.

Enumeration of different selections becomes somewhat more complicated if the same item can be chosen more than once ("sampling with replacement"). Suppose that a customer is given two (identical) coupons, which allow the customer to go to one of several shops and buy two items at a discount at the shop. Buying the same item twice is allowed. For simplicity suppose that there are only two shops A,B and they sell only two different items {A1, A2} and {B1, B2}, respectively. A person may buy A1 four times, by going to the shop A twice and buying A1 twice each time. Or a person may buy each of A1, A2, B1, B2 once. Note that in this scheme it is not possible to buy three items from shop A and 1 item from shop B. Again we can think of a statistical model that the relative popularity of selections is explained entirely by the attractiveness of each item. This corresponds to Example 3.4 below, where A1 $= u_1^{(1)}$, A2 $= u_2^{(1)}$, B1 $= u_1^{(2)}$, B2 $= u_2^{(2)}$.

Note that in the above examples we can also consider recursive nesting of subgroups.

3 Nested configurations

In this section, we introduce an effective method to construct semigroup rings whose toric ideals have quadratic Gröbner bases.

Let A be a configuration of a polynomial ring $K[\mathbf{t}] = K[t_1, \ldots, t_d]$ with d variables. For each $i = 1, 2, \ldots, d$, let $B_i = \{m_1^{(i)}, \ldots, m_{\lambda_i}^{(i)}\}$ be a configuration of a polynomial ring $K[\mathbf{u}^{(i)}] = K[u_1^{(i)}, \ldots, u_{\mu_i}^{(i)}]$ with μ_i variables. The *nested configuration* arising from A and $B_1, ..., B_d$ is the configuration

$$A(B_1, \ldots, B_d) := \left\{ m_{j_1}^{(i_1)} \cdots m_{j_r}^{(i_r)} \;\middle|\; 1 \leq r \in \mathbb{Z},\ t_{i_1} \cdots t_{i_r} \in A, \quad 1 \leq j_k \leq \lambda_{i_k} \right\}$$

of a polynomial ring $K[\mathbf{u}^{(1)}, \ldots, \mathbf{u}^{(d)}]$.

Example 3.1. If $B_i = \left\{ m_1^{(i)} \right\}$ for all $1 \leq i \leq d$, then we have $K[A(B_1, \ldots, B_d)] \simeq K[A]$.

Example 3.2. Let $A = \{t_1^r\}$ and let $B_1 = \{u_1, \ldots, u_\lambda\}$ be the set of variables. Then $K[A(B_1)]$ is rth Veronese subring of the polynomial ring $K[B_1] = K[u_1, \ldots, u_\lambda]$.

Example 3.3. Let $A = \{t_1 t_2\}$ and let $B_1 = \{u_1^{(1)}, \ldots, u_{\lambda_1}^{(1)}\}$ and $B_2 = \{u_1^{(2)}, \ldots, u_{\lambda_2}^{(2)}\}$ be the sets of variables. Then $K[A(B_1, B_2)]$ is the Segre product of the polynomial rings $K[B_1] = K[u_1^{(1)}, \ldots, u_{\lambda_1}^{(1)}]$ and $K[B_2] = K[u_1^{(2)}, \ldots, u_{\lambda_2}^{(2)}]$.

Let η be the cardinality of $A(B_1, \ldots, B_d)$ and set $A(B_1, \ldots, B_d) = \{M_1, \ldots, M_\eta\}$. Let

$$\begin{aligned} K[\mathbf{x}] &= K[x_{M_1}, \ldots, x_{M_\eta}] \\ K[\mathbf{y}] &= K[\{y_{i_1 \cdots i_r}\}_{1 \leq r \in \mathbb{Z},\ i_1 \leq \cdots \leq i_r,\ t_{i_1} \cdots t_{i_r} \in A}] \\ K\left[\mathbf{z}^{(i)}\right] &= K\left[z_1^{(i)}, \ldots, z_{\lambda_i}^{(i)}\right] \qquad (i = 1, 2, \ldots, d) \end{aligned}$$

be polynomial rings. The toric ideal I_A is the kernel of the homomorphism $\pi_0 : K[\mathbf{y}] \longrightarrow K[\mathbf{t}]$ defined by setting $\pi_0(y_{i_1 \cdots i_r}) = t_{i_1} \cdots t_{i_r}$. The toric ideal I_{B_i} is the kernel of the

homomorphism $\pi_i : K[\mathbf{z}^{(i)}] \longrightarrow K[\mathbf{u}^{(i)}]$ defined by setting $\pi_i(z_j^{(i)}) = m_j^{(i)}$. The toric ideal $I_{A(B_1,\ldots,B_d)}$ is the kernel of the homomorphism $\pi : K[\mathbf{x}] \longrightarrow K[\mathbf{u}^{(1)},\ldots,\mathbf{u}^{(d)}]$ defined by setting $\pi(x_M) = M$.

Let $\mathcal{G}_i$ be a Gröbner basis of I_{B_i} with respect to a monomial order $<_i$ for $1 \le i \le d$. For each $M \in A(B_1,\ldots,B_d)$, the expression $M = m_{j_1}^{(i_1)} \cdots m_{j_r}^{(i_r)}$ is called *standard* if

$$\prod_{i_k = j,\quad 1\le k\le r} z_{j_k}^{(i_k)}$$

is a standard monomial with respect to $\mathcal{G}_j$ for all $1 \le j \le d$.

Example 3.4. Let $A = \{t_1^2, t_1t_2, t_2^2\}$,

$$\begin{array}{rcl} B_1 &=& \left\{m_1^{(1)} = \left(u_1^{(1)}\right)^2,\ m_2^{(1)} = u_1^{(1)}u_2^{(1)},\ m_3^{(1)} = \left(u_2^{(1)}\right)^2\right\}, \\ B_2 &=& \left\{m_1^{(2)} = \left(u_1^{(2)}\right)^2,\ m_2^{(2)} = u_1^{(2)}u_2^{(2)},\ m_3^{(2)} = \left(u_2^{(2)}\right)^2\right\}. \end{array}$$

Then $A(B_1,B_2)$ consists of the monomials

$$\left(u_1^{(1)}\right)^4, \left(u_1^{(1)}\right)^3 u_2^{(1)}, \left(u_1^{(1)}\right)^2 \left(u_2^{(1)}\right)^2, u_1^{(1)} \left(u_2^{(1)}\right)^3, \left(u_2^{(1)}\right)^4,$$

$$\left(u_1^{(1)}\right)^2 \left(u_1^{(2)}\right)^2, \left(u_1^{(1)}\right)^2 u_1^{(2)}u_2^{(2)}, \left(u_1^{(1)}\right)^2 \left(u_2^{(2)}\right)^2$$

$$u_1^{(1)}u_2^{(1)} \left(u_1^{(2)}\right)^2, u_1^{(1)}u_2^{(1)}u_1^{(2)}u_2^{(2)}, u_1^{(1)}u_2^{(1)} \left(u_2^{(2)}\right)^2$$

$$\left(u_2^{(1)}\right)^2 \left(u_1^{(2)}\right)^2, \left(u_2^{(1)}\right)^2 u_1^{(2)}u_2^{(2)}, \left(u_2^{(1)}\right)^2 \left(u_2^{(2)}\right)^2$$

$$\left(u_1^{(2)}\right)^4, \left(u_1^{(2)}\right)^3 u_2^{(2)}, \left(u_1^{(2)}\right)^2 \left(u_2^{(2)}\right)^2, u_1^{(2)} \left(u_2^{(2)}\right)^3, \left(u_2^{(2)}\right)^4$$

and, with respect to any monomial order,

$$\begin{array}{rcl} \mathcal{G}_0 &=& \{y_{11}y_{22} - y_{12}^2\}, \\ \mathcal{G}_1 &=& \{z_1^{(1)}z_3^{(1)} - (z_2^{(1)})^2\}, \\ \mathcal{G}_2 &=& \{z_1^{(2)}z_3^{(2)} - (z_2^{(2)})^2\} \end{array}$$

are Gröbner bases of

$$\begin{array}{rcl} I_A &=& \left\langle y_{11}y_{22} - y_{12}^2 \right\rangle, \\ I_{B_1} &=& \left\langle z_1^{(1)}z_3^{(1)} - (z_2^{(1)})^2 \right\rangle, \\ I_{B_2} &=& \left\langle z_1^{(2)}z_3^{(2)} - (z_2^{(2)})^2 \right\rangle, \end{array}$$

respectively. Let $>_0$ be a lexicographic order induced by $y_{11} >_0 y_{12} >_0 y_{22}$ and let $>_i$ a lexicographic order induced by $z_1^{(i)} >_i z_2^{(i)} >_i z_3^{(i)}$ for $i = 1, 2$. For example,

$$M = \left(u_1^{(1)}\right)^2 \left(u_2^{(1)}\right)^2 \in A(B_1, B_2)$$

has two expressions, that is, $M = m_1^{(1)} m_3^{(1)}$ and $M = \left(m_2^{(1)}\right)^2$. Since $z_1^{(1)} z_3^{(1)}$ is not standard and $(z_2^{(1)})^2$ is standard with respect to $\mathcal{G}_1$, $M = m_1^{(1)} m_3^{(1)}$ is not a standard expression and $M = \left(m_2^{(1)}\right)^2$ is a standard expression.

In order to study the relation among I_A, I_{B_i} and $I_{A(B_1,\ldots,B_d)}$, we define homomorphisms

$$\varphi_0 : K[\mathbf{x}] \longrightarrow K[\mathbf{y}] \quad , \quad \varphi_0 \left(x_{m_{j_1}^{(i_1)} \cdots m_{j_r}^{(i_r)}} \right) = y_{i_1 \cdots i_r},$$
$$\varphi_j : K[\mathbf{x}] \longrightarrow K[\mathbf{z}^{(j)}] \quad , \quad \varphi_j \left(x_{m_{j_1}^{(i_1)} \cdots m_{j_r}^{(i_r)}} \right) = \prod_{i_k = j, \ \ 1 \le k \le r} z_{j_k}^{(i_k)},$$

where $m_{j_1}^{(i_1)} \cdots m_{j_r}^{(i_r)}$ is the standard expression defined above. For example,

$$\varphi_1 \left(x_{\left(u_1^{(1)}\right)^2 \left(u_2^{(1)}\right)^2} \right)$$

is not $z_1^{(1)} z_3^{(1)}$ but $(z_2^{(1)})^2$ in Example 3.4. Throughout this paper, we order the monomials of $A(B_1, \ldots, B_d)$ as $A(B_1, \ldots, B_d) = \{M_1, \ldots, M_\eta\}$ where

$$\pi_0 \circ \varphi_0(x_{M_1}) \ge_{lex} \cdots \ge_{lex} \pi_0 \circ \varphi_0(x_{M_\eta}) \qquad (*)$$

with respect to the lexicographic order $<_{lex}$ induced by $t_1 > \cdots > t_d$.

Lemma 3.5. *Let f be a binomial in $K[\mathbf{x}]$. Then the following conditions are equivalent:*

(i) $f \in I_{A(B_1,\ldots,B_d)}$;

(ii) $\varphi_i(f) \in I_{B_i}$ *for all* $1 \le i \le d$.

Moreover, if the above conditions hold, then we have $\varphi_0(f) \in I_A$.

Proof. Let $f = \mathbf{x}^\alpha - \mathbf{x}^\beta$. Since $\pi(x_M) = M = \prod_{j=1}^d \pi_j \circ \varphi_j(x_M)$ and each $\pi_j \circ \varphi_j(x_M)$ belongs to $K[\mathbf{u}^{(j)}]$, we have

$$\begin{aligned} f \in I_{A(B_1,\ldots,B_d)} = \ker(\pi) &\iff \pi(\mathbf{x}^\alpha) = \pi(\mathbf{x}^\beta) \\ &\iff \prod_{j=1}^d \pi_j \circ \varphi_j(\mathbf{x}^\alpha) = \prod_{j=1}^d \pi_j \circ \varphi_j(\mathbf{x}^\beta) \\ &\iff \pi_j \circ \varphi_j(\mathbf{x}^\alpha) = \pi_j \circ \varphi_j(\mathbf{x}^\beta) \ \text{ for all } 1 \le j \le d \\ &\iff \pi_j \circ \varphi_j(f) = 0 \ \text{ for all } 1 \le j \le d \\ &\iff \varphi_j(f) \in \ker(\pi_j) = I_{B_j} \ \text{ for all } 1 \le j \le d. \end{aligned}$$

Thus (i) $\iff$ (ii) holds.

Recall that I_{B_i} is homogeneous for all $1 \le i \le d$. If $M = m_{j_1}^{(i_1)} \cdots m_{j_r}^{(i_r)}$, then

$$\pi_0 \circ \varphi_0(x_M) = t_{i_1} \cdots t_{i_r} = \prod_{j=1}^{d} t_j^{\deg(\varphi_j(x_M))}.$$

Hence we have

$$\begin{aligned} & \varphi_j(f) \in I_{B_j} \quad \text{for all } 1 \le j \le d \\ \Longrightarrow\ & \deg(\varphi_j(\mathbf{x}^\alpha)) = \deg(\varphi_j(\mathbf{x}^\beta)) \quad \text{for all } 1 \le j \le d \\ \Longrightarrow\ & \pi_0 \circ \varphi_0(\mathbf{x}^\alpha) = \prod_{j=1}^{d} t_j^{\deg(\varphi_j(\mathbf{x}^\alpha))} = \prod_{j=1}^{d} t_j^{\deg(\varphi_j(\mathbf{x}^\beta))} = \pi_0 \circ \varphi_0(\mathbf{x}^\beta) \\ \Longrightarrow\ & \pi_0 \circ \varphi_0(f) = 0 \\ \Longrightarrow\ & \varphi_0(f) \in \ker(\pi_0) = I_A \end{aligned}$$

as desired. $\square$

Theorem 3.6. *Suppose that the toric ideals I_A, $I_{B_1}, \ldots, I_{B_d}$ possess quadratic Gröbner bases $\mathcal{G}_0$, $\mathcal{G}_1, \ldots, \mathcal{G}_d$ with respect to $<_0$, $<_1, \ldots, <_d$ respectively. Then the toric ideal $I_{A(B_1,\ldots,B_d)}$ possesses the reduced Gröbner basis $\mathcal{G}$ consisting of all binomials of the form $x_{M_\alpha} x_{M_\beta} - x_{M_\gamma} x_{M_\delta}$ where $M_\gamma = m_{j_1}^{(i_1)} m_{j_2}^{(i_2)} \cdots m_{j_r}^{(i_r)}$, $M_\delta = m_{\ell_1}^{(k_1)} m_{\ell_2}^{(k_2)} \cdots m_{\ell_s}^{(k_s)}$ with $\gamma \le \delta$ and*

$$\varphi_j(x_{M_\alpha} x_{M_\beta}) \xrightarrow{\mathcal{G}_j} \varphi_j(x_{M_\gamma} x_{M_\delta}) \qquad \text{for each } j = 0, 1, \ldots, d, \tag{1}$$

$$i_\lambda = k_\mu \implies j_\lambda \le \ell_\mu \qquad \text{for } 1 \le \lambda \le r,\ 1 \le \mu \le s. \tag{2}$$

The initial monomial of $x_{M_\alpha} x_{M_\beta} - x_{M_\gamma} x_{M_\delta}$ is $x_{M_\alpha} x_{M_\beta}$.

Example 3.4 (continued). The reduced Gröbner basis $\mathcal{G}$ in the statement of Theorem 3.6 consists of 105 binomials. For example,

$$\begin{array}{rcl} x_{m_2^{(1)} m_2^{(1)}} x_{m_2^{(2)} m_2^{(2)}} & - & {x_{m_2^{(1)} m_2^{(2)}}}^2 \\ x_{m_1^{(1)} m_1^{(1)}} x_{m_3^{(1)} m_2^{(2)}} & - & x_{m_1^{(1)} m_2^{(1)}} x_{m_2^{(1)} m_2^{(2)}} \\ x_{m_2^{(1)} m_2^{(1)}} x_{m_1^{(1)} m_2^{(2)}} & - & x_{m_1^{(1)} m_2^{(1)}} x_{m_2^{(1)} m_2^{(2)}} \\ {x_{m_1^{(1)} m_2^{(1)}}}^2 & - & x_{m_1^{(1)} m_1^{(1)}} x_{m_2^{(1)} m_2^{(1)}} \end{array}$$

belong to $\mathcal{G}$ and the initial monomial is the first monomial for each binomial.

Proof of Theorem 3.6. Let $x_{M_\alpha} x_{M_\beta} - x_{M_\gamma} x_{M_\delta} \in \mathcal{G}$. Thanks to the condition (1) above, we have $\varphi_j(x_{M_\alpha} x_{M_\beta}) - \varphi_j(x_{M_\gamma} x_{M_\delta}) \in I_{B_j}$ for all $1 \le j \le d$. By virtue of Lemma 3.5, we have $x_{M_\alpha} x_{M_\beta} - x_{M_\gamma} x_{M_\delta} \in I_{A(B_1,\ldots,B_d)}$. Thus $\mathcal{G}$ is a subset of $I_{A(B_1,\ldots,B_d)}$.

Since the reduction relation modulo Gröbner bases are Noetherian, [Stu, Theorem 3.12] guarantees that there exists a monomial order such that the monomial $x_{M_\alpha} x_{M_\beta}$ is the initial monomial for each $x_{M_\alpha} x_{M_\beta} - x_{M_\gamma} x_{M_\delta} \in \mathcal{G}$.

Suppose that there exists a binomial $0 \neq u - v \in I_{A(B_1,\ldots,B_d)}$ such that neither u nor v is divided by the initial monomial of any binomial in $\mathcal{G}$. By virtue of Lemma 1, we have $\varphi_0(u) - \varphi_0(v) \in I_A$ and $\varphi_i(u) - \varphi_i(v) \in I_{B_i}$ for all $1 \leq i \leq d$. Hence $\varphi_i(u) - \varphi_i(v) \xrightarrow{\mathcal{G}_i} 0$ for all $0 \leq i \leq d$. Moreover, since neither u nor v is divided by the initial monomial of any binomial in $\mathcal{G}$, we have $\varphi_i(u) \xrightarrow{\mathcal{G}_i} \varphi_i(u)$ and $\varphi_i(v) \xrightarrow{\mathcal{G}_i} \varphi_i(v)$. Thus $\varphi_i(u) = \varphi_i(v)$ for all $0 \leq i \leq d$. By virtue of $\varphi_0(u) = \varphi_0(v)$ and our convention $(*)$,

$$\begin{aligned} u &= x_{M_{\ell_1}} x_{M_{\ell_2}} \cdots x_{M_{\ell_p}}, \\ v &= x_{M_{\ell'_1}} x_{M_{\ell'_2}} \cdots x_{M_{\ell'_p}}, \end{aligned}$$

where $1 \leq \ell_1 \leq \cdots \leq \ell_p \leq \eta$, $1 \leq \ell'_1 \leq \cdots \leq \ell'_p \leq \eta$ and

$$\begin{aligned} M_{\ell_\xi} &= m_{j_{\xi,1}}^{(i_{\xi,1})} m_{j_{\xi,2}}^{(i_{\xi,2})} \cdots m_{j_{\xi,r_\xi}}^{(i_{\xi,r_\xi})}, \\ M_{\ell'_\xi} &= m_{k_{\xi,1}}^{(i_{\xi,1})} m_{k_{\xi,2}}^{(i_{\xi,2})} \cdots m_{k_{\xi,r_\xi}}^{(i_{\xi,r_\xi})}. \end{aligned}$$

Since $\varphi_j(u) = \varphi_j(v)$ for all $1 \leq j \leq d$, we have

$$\prod_{i_{\xi,q}=j,\ 1 \leq \xi \leq p,\ 1 \leq q \leq r_\xi} z_{j_{\xi,q}}^{(i_{\xi,q})} = \prod_{i_{\xi,q}=j,\ 1 \leq \xi \leq p,\ 1 \leq q \leq r_\xi} z_{k_{\xi,q}}^{(i_{\xi,q})}.$$

Thanks to the condition (2), we have

$$i_{\xi,q} = i_{\xi',q'},\ \xi < \xi' \Longrightarrow j_{\xi,q} \leq j_{\xi',q'},$$

$$i_{\xi,q} = i_{\xi',q'},\ \xi < \xi' \Longrightarrow k_{\xi,q} \leq k_{\xi',q'}.$$

Hence $M_{\ell_\xi} = M_{\ell'_\xi}$ for all $1 \leq \xi \leq p$. Thus we have $u = v$ and this is a contradiction. □

Example 3.7. In the definition of a nested configuration, we assumed that each B_i and B_j have no common variable. If B_i and B_j have a common variable for some $1 \leq i < j \leq \lambda$, then Theorem 3.6 does not hold in general. For example, if $A = \{t_1t_4, t_2t_5, t_3t_6\}$, $B_1 = \{u_1\}$, $B_2 = \{u_2\}$, $B_3 = \{u_3\}$, $B_4 = \{v_1, v_2\}$, $B_5 = \{v_2, v_3\}$ and $B_6 = \{v_1, v_3\}$, then $I_{A(B_1,\ldots,B_6)}$ is a principal ideal generated by a binomial of degree 3.

4 Nested configurations arising from Segre–Veronese configurations

A typical class of semigroup rings whose toric ideal possesses a quadratic initial ideal is *algebras of Segre–Veronese type* defined in [OH2, AHOT]. Fix integers $\tau \geq 2$ and n and sets of integers $\mathbf{b} = \{b_1, \ldots, b_n\}$, $\mathbf{c} = \{c_1, \ldots, c_n\}$, $\mathbf{p} = \{p_1, \ldots, p_n\}$ and $\mathbf{q} = \{q_1, \ldots, q_n\}$ such that

(i) $0 \le c_i \le b_i$ for all $1 \le i \le n$;

(ii) $1 \le p_i \le q_i \le d$ for all $1 \le i \le n$.

Let $A_{\tau,\mathbf{b},\mathbf{c},\mathbf{r},\mathbf{s}} \subset K[t_1, \ldots, t_d]$ denote the set of all monomials $\prod_{j=1}^{d} {t_j}^{f_j}$ such that

(i) $\sum_{j=1}^{d} f_j = \tau$.

(ii) $c_i \le \sum_{j=p_i}^{q_i} f_j \le b_i$ for all $1 \le i \le n$.

Then the affine semigroup ring $K[A_{\tau,\mathbf{b},\mathbf{c},\mathbf{r},\mathbf{s}}]$ is called an *algebra of Segre–Veronese type*.

Several popular classes of semigroup rings are algebras of Segre–Veronese type. If $n = 2$, $\tau = 2$, $b_1 = b_2 = c_1 = c_2 = 1$, $p_1 = 1$, $p_2 = q_1 + 1$ and $q_2 = d$, then the affine semigroup ring $K[A_{\tau,\mathbf{b},\mathbf{c},\mathbf{r},\mathbf{s}}]$ is the Segre product of polynomial rings $K[t_1, \ldots, t_{q_1}]$ and $K[t_{q_1+1}, \ldots, t_d]$. On the other hand, if $n = d$, $p_i = q_i = i$, $b_i = \tau$ and $c_i = 0$ for all $1 \le i \le n$, then the affine semigroup ring $K[A_{\tau,\mathbf{b},\mathbf{c},\mathbf{r},\mathbf{s}}]$ is the classical τth Veronese subring of the polynomial ring $K[t_1, \ldots, t_d]$. Moreover, if $n = d$, $p_i = q_i = i$, $b_i = 1$ and $c_i = 0$ for all $1 \le i \le n$, then the affine semigroup ring $K[A_{\tau,\mathbf{b},\mathbf{c},\mathbf{r},\mathbf{s}}]$ is the τth squarefree Veronese subring of the polynomial ring $K[t_1, \ldots, t_d]$. In addition, algebras of Veronese type (i.e., $n = d$, $p_i = q_i = i$ and $c_i = 0$ for all $1 \le i \le n$) are studied in [DeHi] and [Stu].

Let $K[X]$ denote the polynomial ring with the set of variables

$$\left\{ x_{j_1 j_2 \cdots j_\tau} \;\middle|\; 1 \le j_1 \le j_2 \le \cdots \le j_\tau \le d, \; \prod_{k=1}^{\tau} t_{j_k} \in A_{\tau,\mathbf{b},\mathbf{c},\mathbf{r},\mathbf{s}} \right\}.$$

The toric ideal $I_{A_{\tau,\mathbf{b},\mathbf{c},\mathbf{r},\mathbf{s}}}$ is the kernel of the surjective homomorphism $\pi : K[X] \longrightarrow K[A_{\tau,\mathbf{b},\mathbf{c},\mathbf{r},\mathbf{s}}]$ defined by $\pi(x_{j_1 j_2 \cdots j_\tau}) = \prod_{k=1}^{\tau} t_{j_k}$. A monomial $x_{\ell_1 \ell_2 \cdots \ell_\tau} x_{m_1 m_2 \cdots m_\tau} \cdots x_{n_1 n_2 \cdots n_\tau}$ is called *sorted* if we have

$$\ell_1 \le m_1 \le \cdots \le n_1 \le \ell_2 \le m_2 \le \cdots \le n_2 \le \cdots \le \ell_\tau \le m_\tau \le \cdots \le n_\tau.$$

Let $\mathrm{sort}(\cdot)$ denote the operator which takes any string over the alphabet $\{1, 2, \ldots, d\}$ and sorts it into weakly increasing order.

The squarefree quadratic Gröbner basis of the toric ideal $I_{A_{\tau,\mathbf{b},\mathbf{c},\mathbf{r},\mathbf{s}}}$ is given as follows.

Theorem 4.1 ([Stu, OH2, AHOT]). *Work with the same notation as above. Let $\mathcal{G}$ be the set of all binomials*

$$x_{\ell_1 \ell_2 \cdots \ell_\tau} x_{m_1 m_2 \cdots m_\tau} - x_{n_1 n_3 \cdots n_{2\tau-1}} x_{n_2 n_4 \cdots n_{2\tau}}$$

where

$$\mathrm{sort}(\ell_1 m_1 \ell_2 m_2 \cdots \ell_\tau m_\tau) = n_1 n_2 \cdots n_{2\tau}.$$

Then there exists a monomial order on $K[X]$ such that $\mathcal{G}$ is the reduced Gröbner basis of the toric ideal $I_{A_{\tau,\mathbf{b},\mathbf{c},\mathbf{r},\mathbf{s}}}$. The initial ideal is generated by squarefree quadratic (nonsorted) monomials.

By virtue of Theorem 3.6, if all of $A, B_1, \dots, B_d$ are arising from Segre–Veronese configurations, then the toric ideal of the nested configuration $A(B_1, \dots, B_d)$ possesses a quadratic Gröbner basis. Although the following Gröbner basis is different from that Theorem 3.6 guarantee, the proof is similar.

Theorem 4.2. *If $K[A]$ is an algebra of Segre–Veronese type, and if the toric ideals $I_{B_1}, \dots, I_{B_d}$ possess the reduced Gröbner basis $\mathcal{G}_i$, then the toric ideal $I_{A(B_1,\dots,B_d)}$ possesses a quadratic Gröbner basis $\mathcal{G}$ consisting of all binomial of the form*

$$\mathbf{x}^\alpha - \mathbf{x}^\beta = x_{m_{j_1}^{(i_1)} m_{j_3}^{(i_3)} \cdots m_{j_{2r-1}}^{(i_{2r-1})}} x_{m_{j_2}^{(i_2)} m_{j_4}^{(i_4)} \cdots m_{j_{2r}}^{(i_{2r})}} - x_{m_{\ell_1}^{(k_1)} m_{\ell_3}^{(k_3)} \cdots m_{\ell_{2r-1}}^{(k_{2r-1})}} x_{m_{\ell_2}^{(k_2)} m_{\ell_4}^{(k_4)} \cdots m_{\ell_{2r}}^{(k_{2r})}}$$

where $\mathbf{x}^\alpha$ is the initial monomial and

$$\begin{aligned} k_1 k_2 \cdots k_{2r} &= \operatorname{sort}(i_1 i_2 \cdots i_{2r}), & (3)\\ \varphi_j(\mathbf{x}^\alpha) &\xrightarrow{\mathcal{G}_j} \varphi_j(\mathbf{x}^\beta) \quad \textit{for each } 1 \le j \le d, & (4)\\ k_i = k_{i+1} &\Longrightarrow \ell_i \le \ell_{i+1} \quad (1 \le i \le 2r-1). & (5) \end{aligned}$$

Moreover, if the initial ideal of I_{B_i} is squarefree for all i, then the initial ideal of $I_{A(B_1,\dots,B_d)}$ is squarefree.

Proof. By virtue of Lemma 3.5, $\mathcal{G}$ is a subset of $I_{A(B_1,\dots,B_d)}$. Since both the sorting operation and the reduction relation modulo Gröbner bases are Noetherian, there exists a monomial order such that the first monomial $\mathbf{x}^\alpha$ is the initial monomial.

Suppose that there exists a binomial $0 \ne u - v \in I_{A(B_1,\dots,B_d)}$ such that neither u nor v is divided by any initial monomial of $\mathcal{G}$. Let

$$\begin{aligned} u &= x_{m_{j_1}^{(i_1)} m_{j_{p+1}}^{(i_{p+1})} \cdots m_{j_{(r-1)p+1}}^{(i_{(r-1)p+1})}} x_{m_{j_2}^{(i_2)} m_{j_{p+2}}^{(i_{p+2})} \cdots m_{j_{(r-1)p+2}}^{(i_{(r-1)p+2})}} \cdots x_{m_{j_p}^{(i_p)} m_{j_{2p}}^{(i_{2p})} \cdots m_{j_{rp}}^{(i_{rp})}},\\ v &= x_{m_{\ell_1}^{(k_1)} m_{\ell_{p+1}}^{(k_{p+1})} \cdots m_{\ell_{(r-1)p+1}}^{(k_{(r-1)p+1})}} x_{m_{\ell_2}^{(k_2)} m_{\ell_{p+2}}^{(k_{p+2})} \cdots m_{\ell_{(r-1)p+2}}^{(k_{(r-1)p+2})}} \cdots x_{m_{\ell_p}^{(k_p)} m_{\ell_{2p}}^{(k_{2p})} \cdots m_{\ell_{rp}}^{(k_{rp})}}. \end{aligned}$$

By virtue of Lemma 3.5, we have

$$\operatorname{sort}(i_1 i_2 \cdots i_{rp}) = \operatorname{sort}(k_1 k_2 \cdots k_{rp}).$$

Thanks to the condition (3), we have

$$i_1 i_2 \cdots i_{rp} = \operatorname{sort}(i_1 i_2 \cdots i_{rp}) = \operatorname{sort}(k_1 k_2 \cdots k_{rp}) = k_1 k_2 \cdots k_{rp}.$$

Hence $i_q = k_q$ for all $1 \le q \le rp$. By virtue of Lemma 3.5, we have $\varphi_j(u) - \varphi_j(v) \in I_{B_j}$ for each $1 \le i \le d$. Since each $\mathcal{G}_i$ consists of quadratic binomials and thanks to the condition (4), $\varphi_j(u) = \varphi_j(v)$ for each $1 \le i \le d$. Hence

$$\prod_{i_q = j,\ \ 1 \le q \le rp} z_{j_q}^{(i_q)} = \prod_{k_q = j,\ \ 1 \le q \le rp} z_{\ell_q}^{(k_q)}. \qquad (6)$$

Thanks to the condition (5) together with (6) above, $j_q = \ell_q$ for all $1 \le q \le rp$. Thus we have $u = v$ and this is a contradiction.

Suppose that the initial ideal of I_{B_i} is squarefree for all i and that ${x_{m_{j_1}^{(i_1)} m_{j_2}^{(i_2)} \cdots m_{j_r}^{(i_r)}}}^2$ belongs to the initial ideal of $I_{A(B_1,\ldots,B_d)}$. Then

$$g = {x_{m_{j_1}^{(i_1)} m_{j_2}^{(i_2)} \cdots m_{j_r}^{(i_r)}}}^2 - x_{m_{\ell_1}^{(k_1)} m_{\ell_3}^{(k_3)} \cdots m_{\ell_{2r-1}}^{(k_{2r-1})}} x_{m_{\ell_2}^{(k_2)} m_{\ell_4}^{(k_4)} \cdots m_{\ell_{2r}}^{(k_{2r})}}$$

belongs to $\mathcal{G}$. Thanks to the condition (3), we have

$$g = {x_{m_{j_1}^{(i_1)} m_{j_2}^{(i_2)} \cdots m_{j_r}^{(i_r)}}}^2 - x_{m_{\ell_1}^{(i_1)} m_{\ell_3}^{(i_2)} \cdots m_{\ell_{2r-1}}^{(i_r)}} x_{m_{\ell_2}^{(i_1)} m_{\ell_4}^{(i_2)} \cdots m_{\ell_{2r}}^{(i_r)}}.$$

Since $g \neq 0$, there exists k such that $j_k \neq \ell_{2k}$. Thanks to the condition (5), we have

$$\varphi_k \left({x_{m_{j_1}^{(i_1)} m_{j_2}^{(i_2)} \cdots m_{j_r}^{(i_r)}}}^2 \right) \neq \varphi_k \left(x_{m_{\ell_1}^{(i_1)} m_{\ell_3}^{(i_2)} \cdots m_{\ell_{2r-1}}^{(i_r)}} x_{m_{\ell_2}^{(i_1)} m_{\ell_4}^{(i_2)} \cdots m_{\ell_{2r}}^{(i_r)}} \right).$$

Hence by the condition (4),

$$\varphi_k \left({x_{m_{j_1}^{(i_1)} m_{j_2}^{(i_2)} \cdots m_{j_r}^{(i_r)}}}^2 \right) = \varphi_k \left(x_{m_{j_1}^{(i_1)} m_{j_2}^{(i_2)} \cdots m_{j_r}^{(i_r)}} \right)^2$$

belongs to the initial ideal of I_{B_k}. Since the initial ideal of I_{B_k} is squarefree,

$$\varphi_k \left(x_{m_{j_1}^{(i_1)} m_{j_2}^{(i_2)} \cdots m_{j_r}^{(i_r)}} \right)$$

belongs to the initial ideal of I_{B_k}. This contradicts that φ_k is defined with a standard expression. Thus the initial ideal of $I_{A(B_1,\ldots,B_d)}$ is squarefree. $\square$

Example 4.3. Let $A = \{t_1^2\}$ and $B_1 = \{u_1, u_2, u_3\}$. Then

$$A(B_1) = \{u_1^2, u_1u_2, u_1u_3, u_2^2, u_2u_3, u_3^2\}.$$

The reduced Gröbner basis in Theorem 3.6 consists of the binomials

$$\begin{array}{rcl}
x_{u_1u_2}^2 & - & x_{u_1^2} x_{u_2^2} \\
x_{u_1u_3}^2 & - & x_{u_1^2} x_{u_3^2} \\
x_{u_2u_3}^2 & - & x_{u_2^2} x_{u_3^2} \\
x_{u_1u_2} x_{u_1u_3} & - & x_{u_1^2} x_{u_2u_3} \\
x_{u_1u_3} x_{u_2^2} & - & x_{u_1u_2} x_{u_2u_3} \\
x_{u_1u_3} x_{u_2u_3} & - & x_{u_1u_2} x_{u_3^2}
\end{array}$$

and the reduced Gröbner basis in Theorem 4.2 consists of the binomials

$$\begin{array}{rcl} x_{u_1^2}x_{u_2^2} & - & x_{u_1u_2}^2 \\ x_{u_1^2}x_{u_3^2} & - & x_{u_1u_3}^2 \\ x_{u_2^2}x_{u_3^2} & - & x_{u_2u_3}^2 \\ x_{u_1^2}x_{u_2u_3} & - & x_{u_1u_2}x_{u_1u_3} \\ x_{u_1u_3}x_{u_2^2} & - & x_{u_1u_2}x_{u_2u_3} \\ x_{u_1u_2}x_{u_3^2} & - & x_{u_1u_3}x_{u_2u_3} \end{array}$$

where the initial monomial of each binomial is the first monomial.

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